INTRODUCTION TO LINEAR ELASTIC FRACTURE MECHANICS

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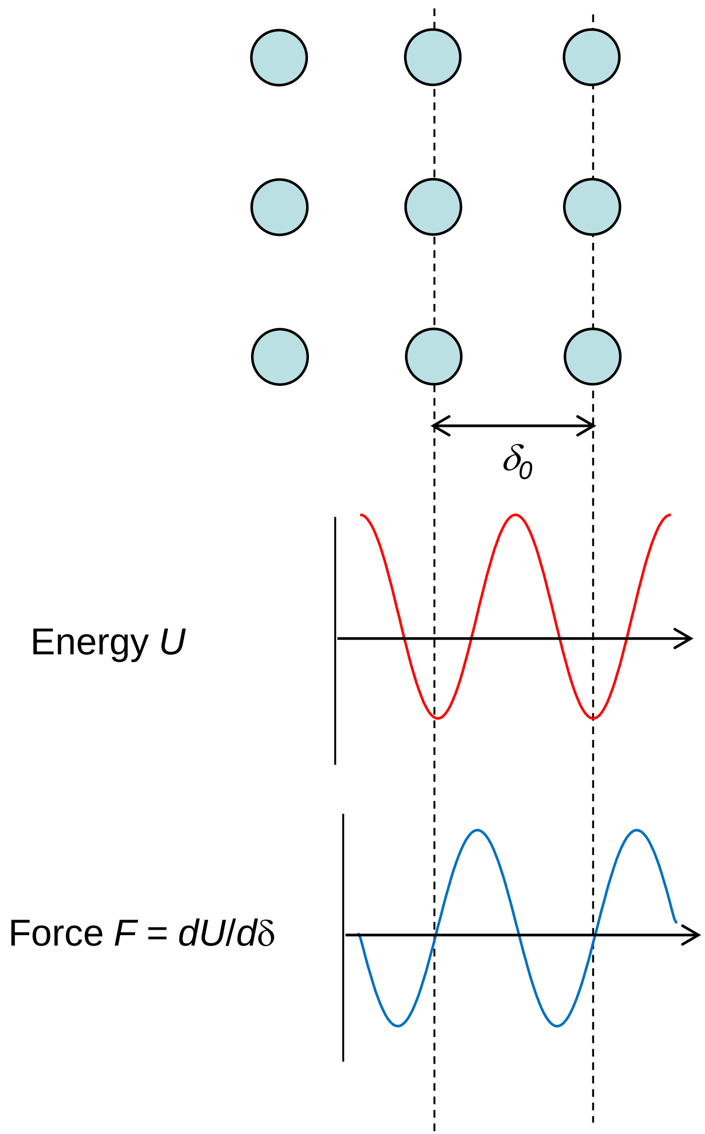
Theoretical strength of a perfectly ordered crystalline lattice



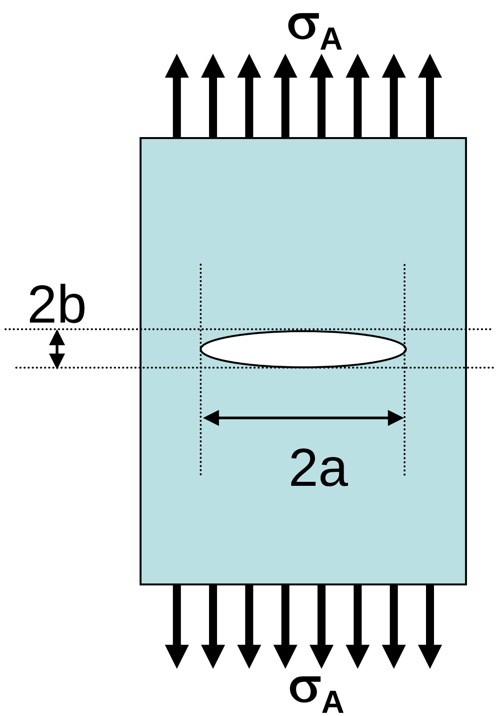
$$\sigma = \sigma_{th} sin\left(\frac{2\pi d}{\delta_0}\right)$$

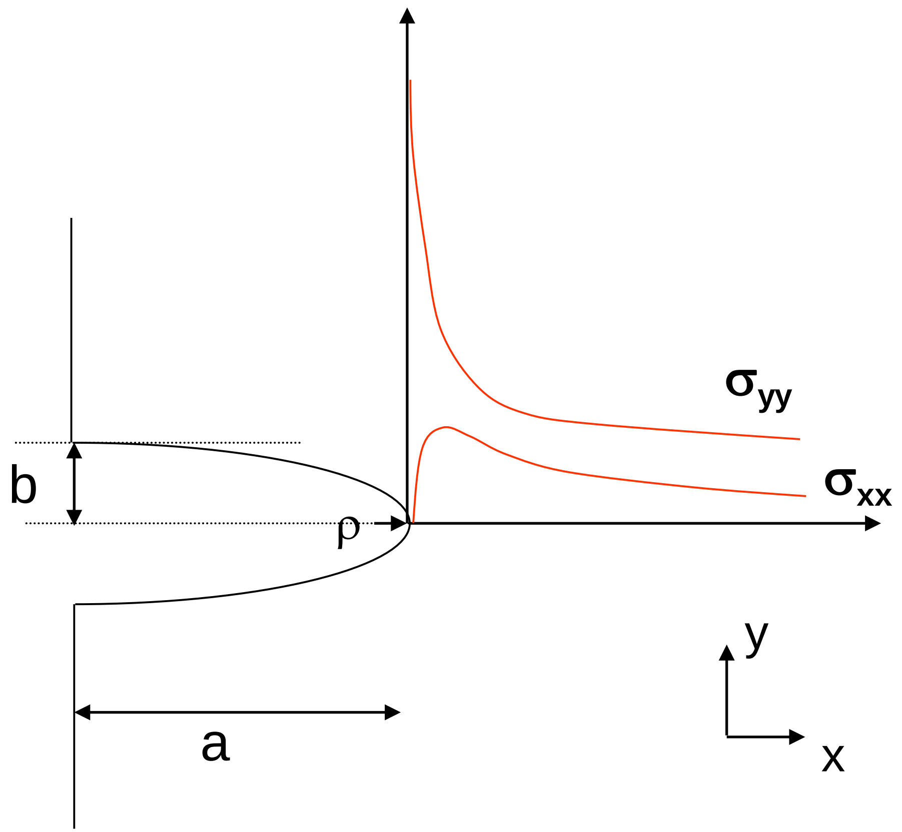
$$\sigma_{th} \approx E/10$$

Stress concentration at the vicinity of an elliptical hole

- Inglis calculation

Uniformly stressed plate

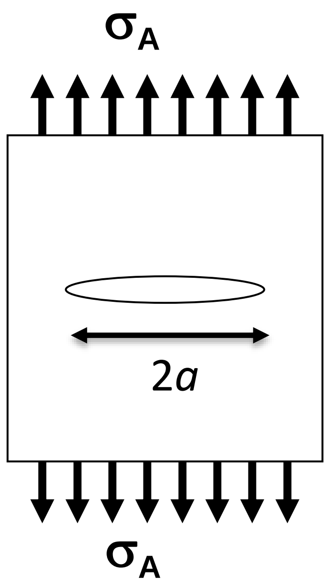




$$\rho = \frac{b^2}{c}$$

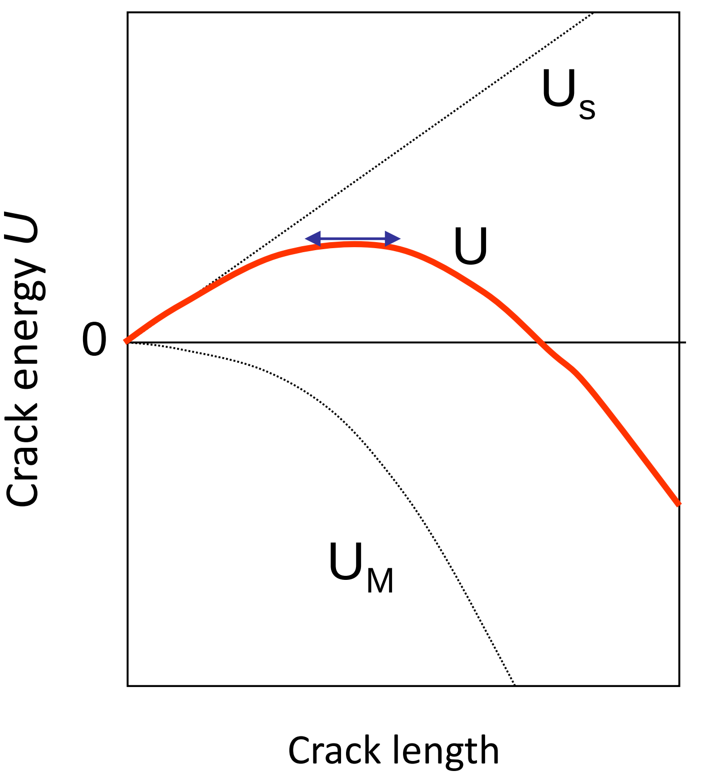
$$\frac{\sigma_c}{\sigma_a} \approx 2\frac{c}{b} = 2\sqrt{\frac{c}{\rho}}$$

Crack in uniform tension



Infinitely narrow elliptical cavity

Remote unifomr tensile stress field σ_A

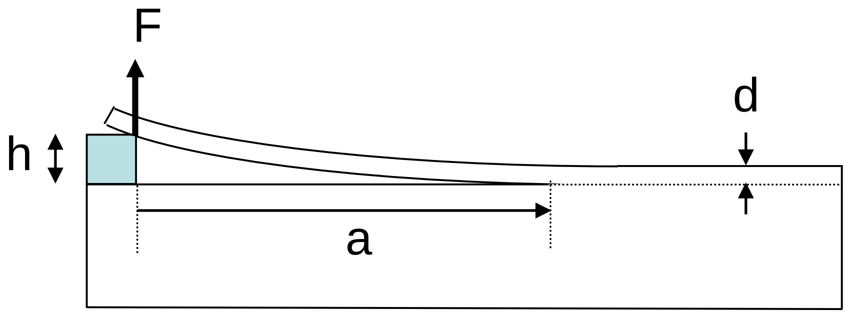


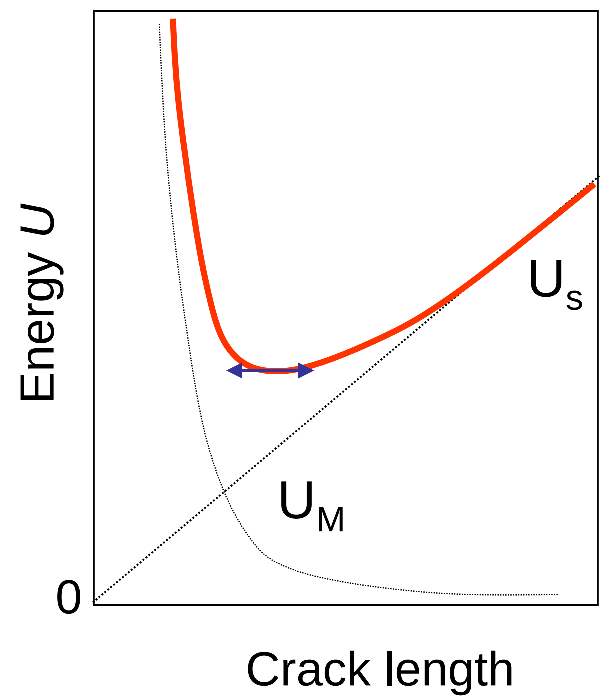
$$U\,(a) = -\pi a^2 \sigma_A^2 / E' + 4a\gamma$$

Unstable configuration

Cleavage by a wedge

- Obreimoff's experiment (cleavage of mica)

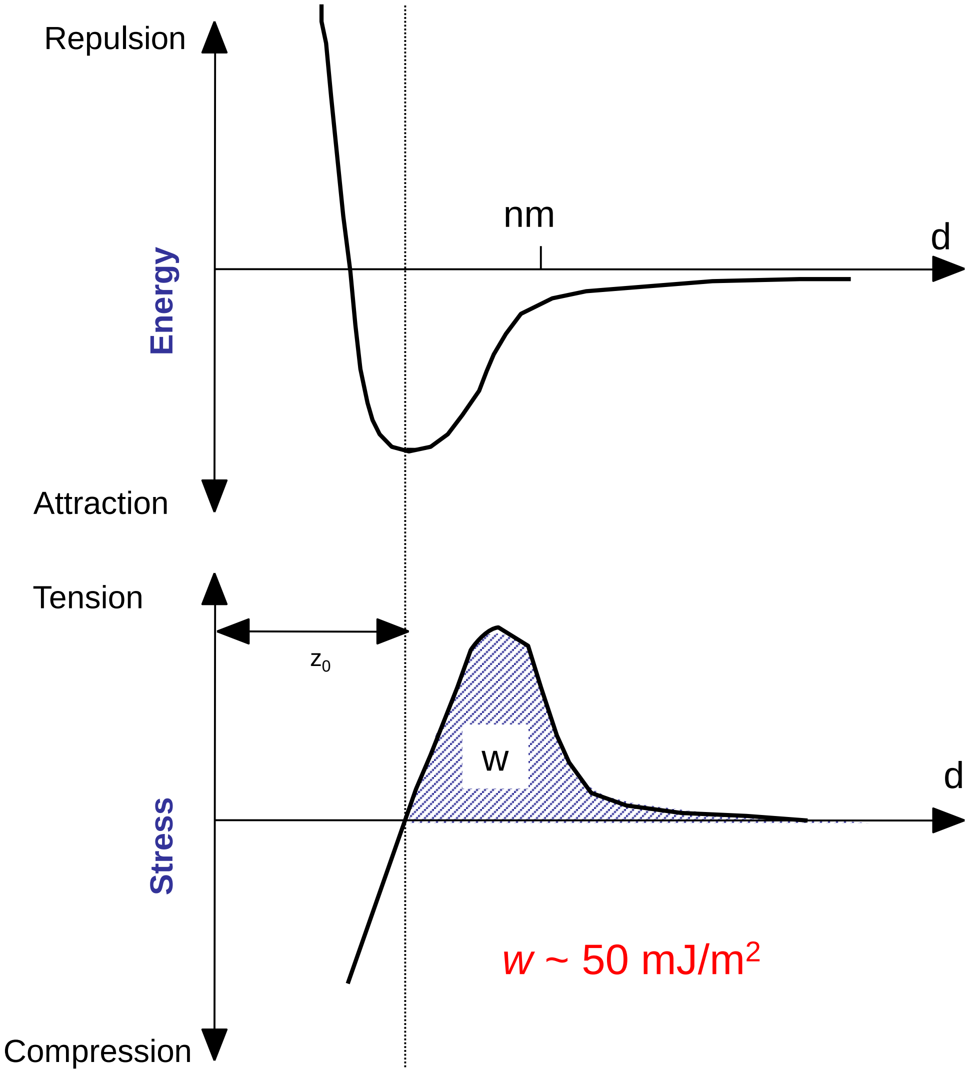


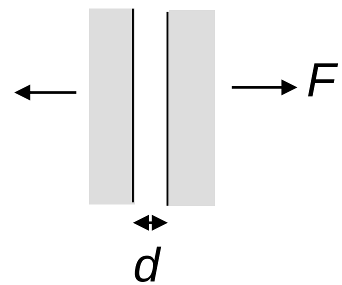


$$U\left(a\right) = \frac{Ed^3h^2}{8c^3} + 2a\gamma$$

Stable configuration

Stress-separation function for two atom planes





Electrostatic interfactions
(Van der Walls,…)

The Dupré work of adhesion:

$$w = \gamma_1 + \gamma_2 - \gamma_{12}$$

Homogeneous solid $w = 2\gamma$

Equilibrium conditions !
No chemistry !

Condition for equilibrium fracture : the Griffith criterion

- Energy balance for crack extension

$$dU = dU_M + dU_S = -GdA + wdA$$

where $G = -\left(\frac{\partial U_E}{\partial A} + \frac{\partial U_P}{\partial A}\right)_P = \left(\frac{\partial U_E}{\partial A}\right)_\delta$ Strain energy release rate

$$\frac{\partial U}{\partial A} = 0 \qquad \boxed{G = w}$$

EQUILIBRIUM condition !!

No dissipative processes !!

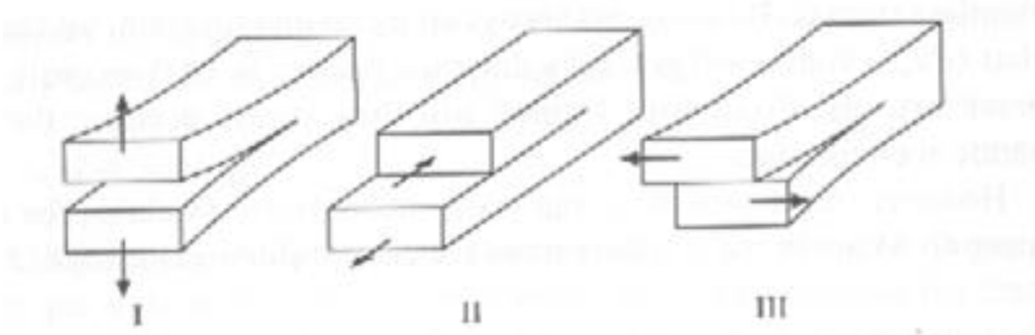
In most practical situations $G_c >> w$

➡ - Energy dissipation at the crack tip (plasticity, viscoelasticity) and/or in the bulk
- Dependence of actual fracture micro-mechanisms at the crack tip (process zone)

……

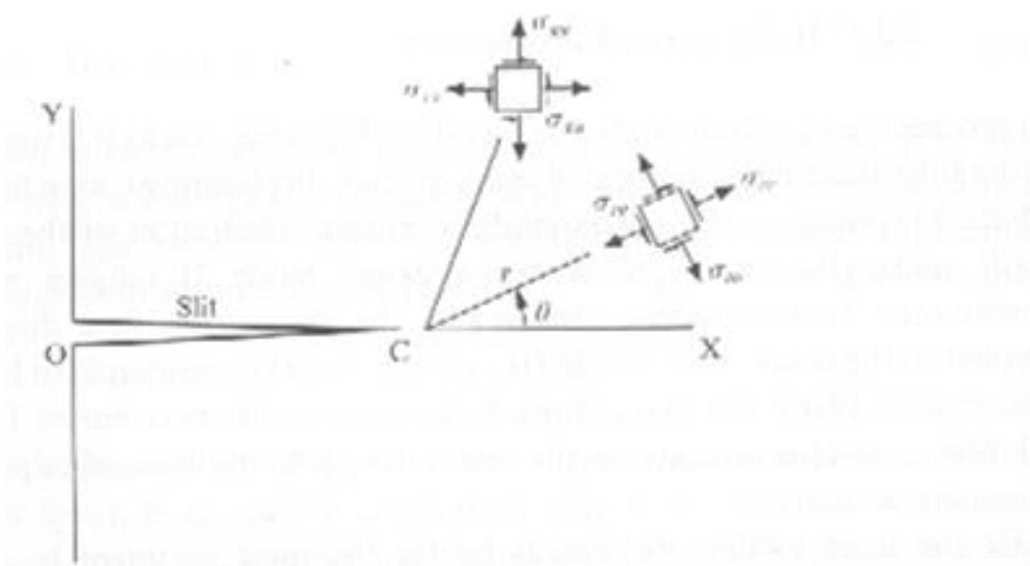
Irwin's approach : Linear elastic crack-tip fields

- The three modes of fracture



I : opening mode
II: sliding mode (in plane shear)
III: tearing mode (out of plane shear)

- Irwin slit-crack tip in rectangular and polar coordinates



Stress-intensity factors K

$$\sigma_{ij} = \frac{K}{\sqrt{2\pi r}} f_{ij}(\theta)$$

$$u_i = \frac{K}{2E}\sqrt{\frac{r}{2\pi}} f_i(\theta)$$

$$K = \chi \sigma \sqrt{a}$$

K=*f* (geometry, applied loading)

Mode I crack loading

$$\kappa = (3-\nu)/(1+\nu), \quad \nu' = 0, \quad \nu'' = \nu, \quad \text{(plane stress)}$$
$$\kappa = (3-4\nu), \quad \nu' = \nu, \quad \nu'' = 0, \quad \text{(plane strain).}$$

Mode I:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \frac{K_I}{(2\pi r)^{1/2}} \begin{Bmatrix} \cos(\theta/2)[1-\sin(\theta/2)\sin(3\theta/2)] \\ \cos(\theta/2)[1+\sin(\theta/2)\sin(3\theta/2)] \\ \sin(\theta/2)\cos(\theta/2)\cos(3\theta/2) \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{r\theta} \end{Bmatrix} = \frac{K_I}{(2\pi r)^{1/2}} \begin{Bmatrix} \cos(\theta/2)[1+\sin^2(\theta/2)] \\ \cos^3(\theta/2) \\ \sin(\theta/2)\cos^2(\theta/2) \end{Bmatrix}$$

$$\sigma_{zz} = \nu'(\sigma_{xx}+\sigma_{yy}) = \nu'(\sigma_{rr}+\sigma_{\theta\theta})$$

$$\sigma_{xz} = \sigma_{yz} = \sigma_{rz} = \sigma_{\theta z} = 0$$

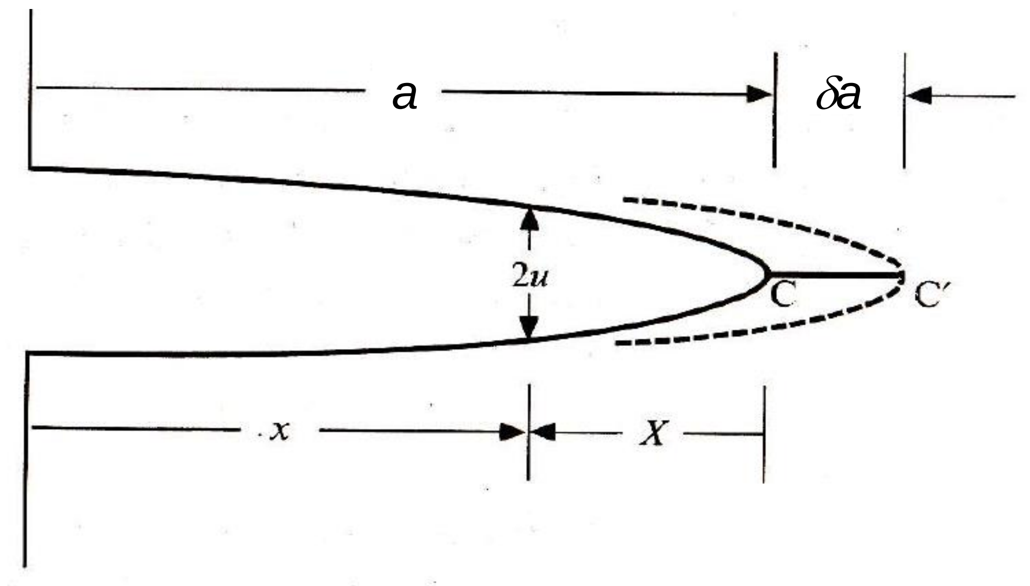
$$\begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \frac{K_I}{2E}\left\{\frac{r}{2\pi}\right\}^{1/2} \begin{Bmatrix} (1+\nu)[(2\kappa-1)\cos(\theta/2)-\cos(3\theta/2)] \\ (1+\nu)[(2\kappa+1)\sin(\theta/2)-\sin(3\theta/2)] \end{Bmatrix}$$

$$\begin{Bmatrix} u_r \\ u_\theta \end{Bmatrix} = \frac{K_I}{2E}\left\{\frac{r}{2\pi}\right\}^{1/2} \begin{Bmatrix} (1+\nu)[(2\kappa-1)\cos(\theta/2)-\cos(3\theta/2)] \\ (1+\nu)[-(2\kappa+1)\sin(\theta/2)+\sin(3\theta/2)] \end{Bmatrix}$$

$$u_z = -(\nu'' z/E)(\sigma_{xx}+\sigma_{yy}) = -(\nu'' z/E)(\sigma_{rr}+\sigma_{\theta\theta}).$$

Equivalence of G and K parameters

- Extension and closure of the crack increment *CC'*



Fixed grip (*u*=const) calculation of the strain-energy release:

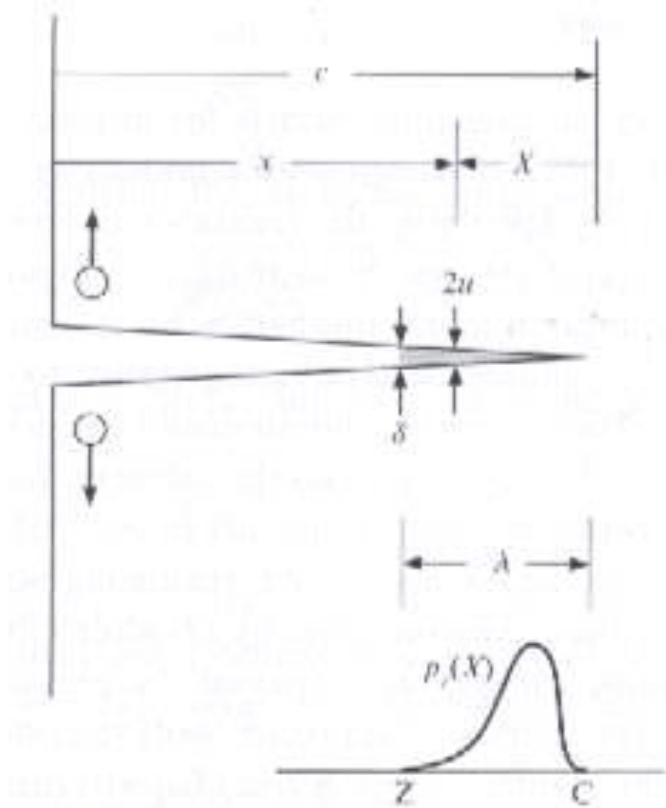
$$\delta U_E = 2\int_{a+\delta a}^{a} 1/2\left(\sigma_{yy}u_y + \sigma_{xy}u_x + \sigma_{zy}u_z\right)dx$$

$$G = -\left(\frac{\partial U_E}{\partial c}\right)_u = K_I^2/E' + K_{II}^2/E' + K_{III}^2(1+\nu)/E$$

$$plane\ strain:\quad E' = E/(1-\nu^2)\ ;\ plane\ stress:\quad E' = E$$

Cohesive zone models

Dugdale, Barenblatt



$\lambda << c << L$

- Non linear cohesive stress p(X,u) acting accross the walls of the slip
- Superposition of the K fields arising from the ‘external’ and ’internal contributions’

Physical origins of the cohesive zone

- Surface forces acting across the crack faces
- Platic deformation at the crack tip (Dugdale model)
- Crack bridging in crazes (polymer fracture)
- ....

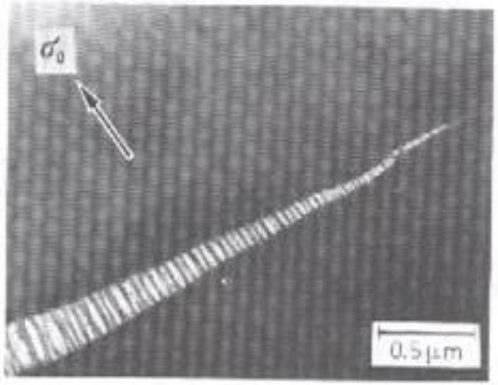


Fig. 9.9. Newly formed craze in thin slice cut from uncrazed bulk polystyrene; craze growing from left to right in a direction perpendicular to that of the uniaxial tensile stress (Courtesy D. Hull [106]).

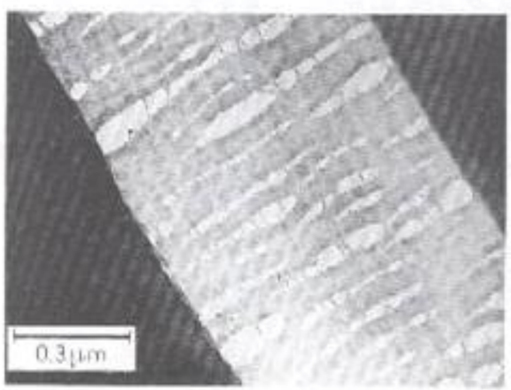


Fig. 9.10. Electron micrograph of the central section of a craze grown as that in Figure 9.9 (Courtesy D. Hull [106]).

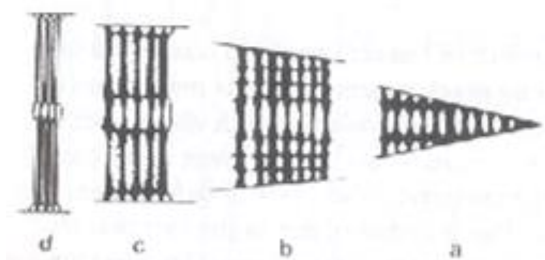
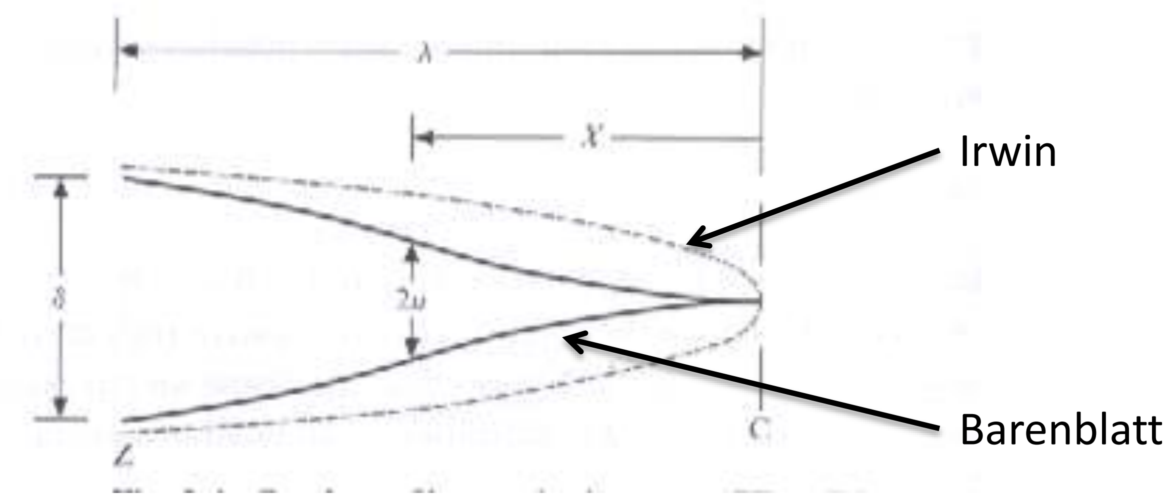


Fig. 9.11. Schematic diagram of the variation in polystyrene craze structure with increasing craze width; the angle at the tip of the craze is exaggerated and the scale in region *d* is larger than that of *a* to *c*. (Courtesy D. Hull [115]).

Barenblatt model: crack-opening profile



Hyp: $p(X,u)_{X\to c} \to 0$

$$u(X)_{X\to c} \propto X^{(}3/2)$$

$$\epsilon_{X\to c} = du/dX \to 0$$

Irwin

$$u(X) \propto X^{1/2}$$

$$\epsilon_{X\to c} \to \infty$$

Path independent integrals about crack tip : The J integral approach

Eshelby, Rice

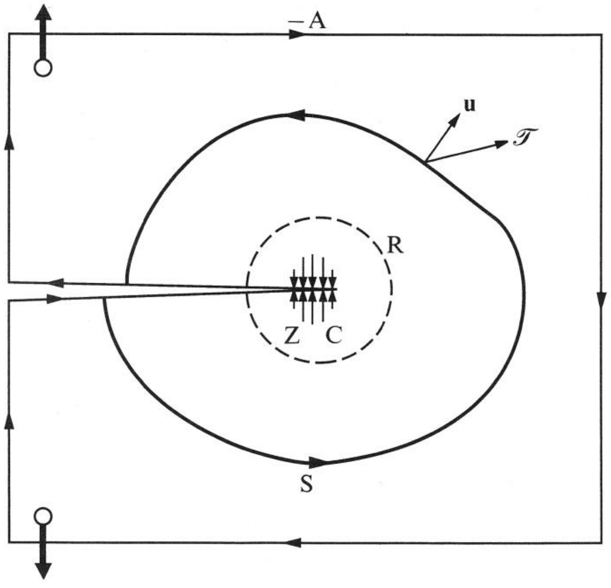


Fig. 3.6. Line integration path S about crack tip C in plane static system, unit thickness. *J*-integral around closed circuit S-A (arrows) is zero. Special *J*-integrals are taken around polar circuit R and cohesion zone CZ.

$$U_M = U_E + U_P = \int_A \Phi dA - \int_S \chi.\mathbf{u}ds$$

$$-dU_M/da = \int_S [\Phi dy - \chi.\left(\partial\mathbf{u}/\partial x\right) ds] = J$$

J integral → holds for any *reversible* deformation response *linear* or non *linear*
→ path independent

Brittle materials : Hertzian fracture / I

Sphere on flat contacts

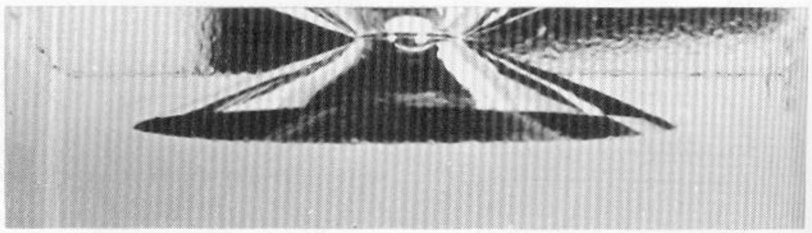


Fig. 8.3. Cone crack in soda-lime glass. Photographed under load ($P = 40$ kN) from cylindrical punch, optical micrograph (block edge length 50 mm). Crack makes angle $\simeq 22°$ to free surface. (After Roesler, F. C. (1956) *Proc. Phys. Soc. Lond.* **B69** 981.)

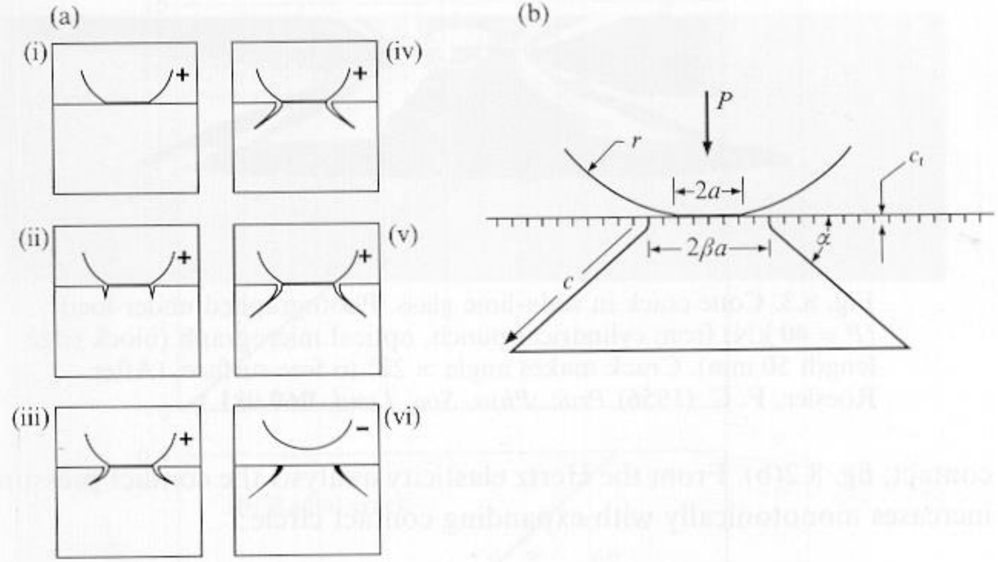


Fig. 8.2. Hertzian cone crack system. (a) Evolution of cone during complete loading (+) and unloading (−) cycle. (b) Geometrical parameters.

(ii) Crack initiation at the edge of the contact

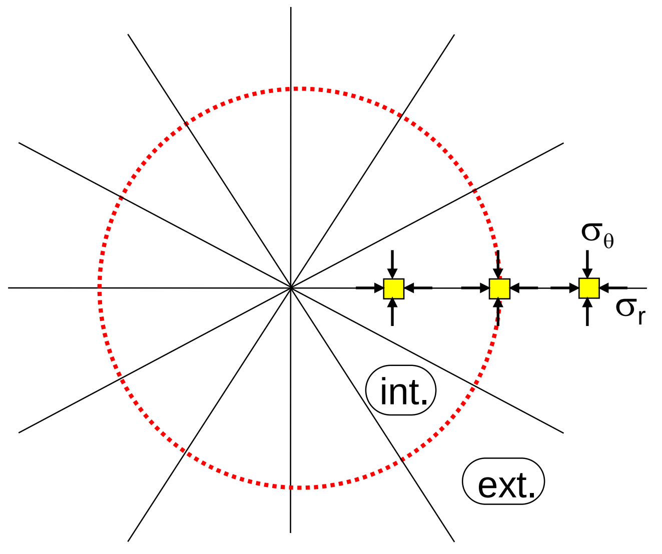
(iii) Stable crack propagation

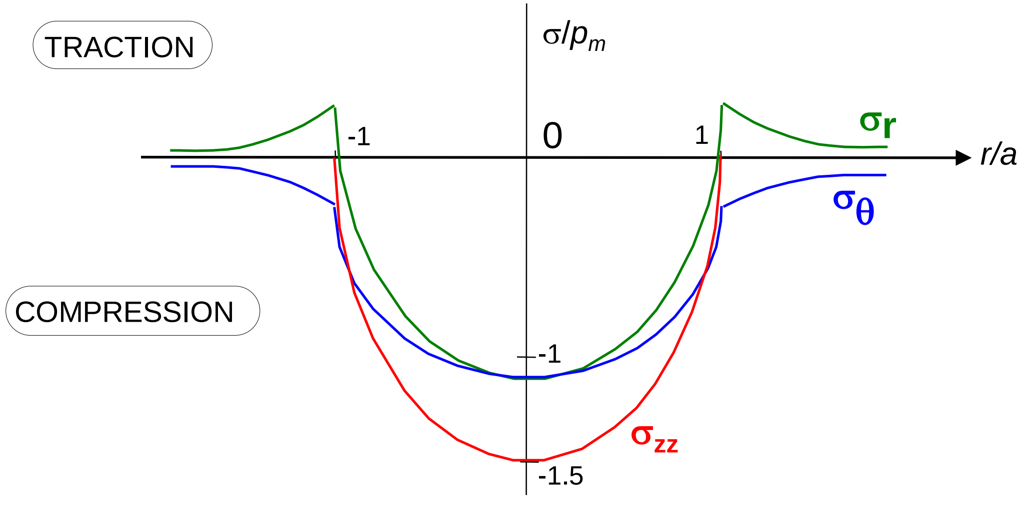
(iv) Instable crack propagation : formation of the Hertzian cone

(v) Stable propagation of the cone

(vi) Crack closing during unloading

Hertz contact : surface tractions





Maximum tensile stress:

$$\boxed{\sigma_R = \frac{1}{2}(1-2\nu)p_m \approx 0.1 p_m}$$

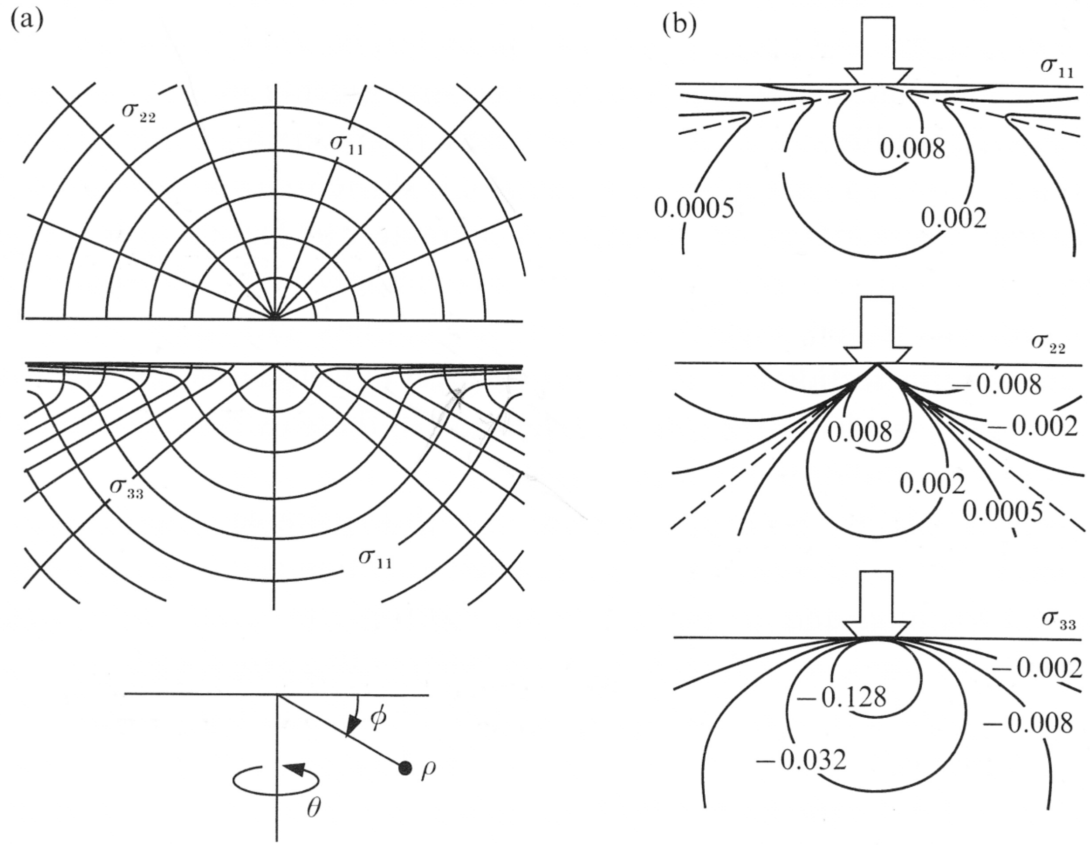
p_m = mean contact pressure

Point surface loading: sub-surface stress fields

Stress trajectories | **Contours**

Half surface

Side view



Side view

Fig. 8.1. Boussinesq field, for principal normal stresses σ_{11}, σ_{22} and σ_{33}. (a) Stress trajectories, half-surface view (top) and side view (bottom). (b) Contours, side view. Unit of contact stress is p_0, contact diameter $2a$ (arrow). Note sharp minimum in $\sigma_{11}(\phi)$ and zero in $\sigma_{22}(\phi)$, dashed lines. Plotted for $\nu = 0.25$. (See Johnson, K. L. (1985) *Contact Mechanics*. Cambridge University Press, Cambridge, Ch. 3.)

Critical load for the propagation of the Hertzian cone crack

- Stress-based approach: $\sigma_R = \frac{1}{2}(1-2\nu)p_m \Rightarrow P_c \propto R^2$

Experimentally $P_c \propto R$!

- Linear elastic fracture mechanics analysis approach

Non homogeneous stress field: the stress intensity factor is evolving as along the crack path

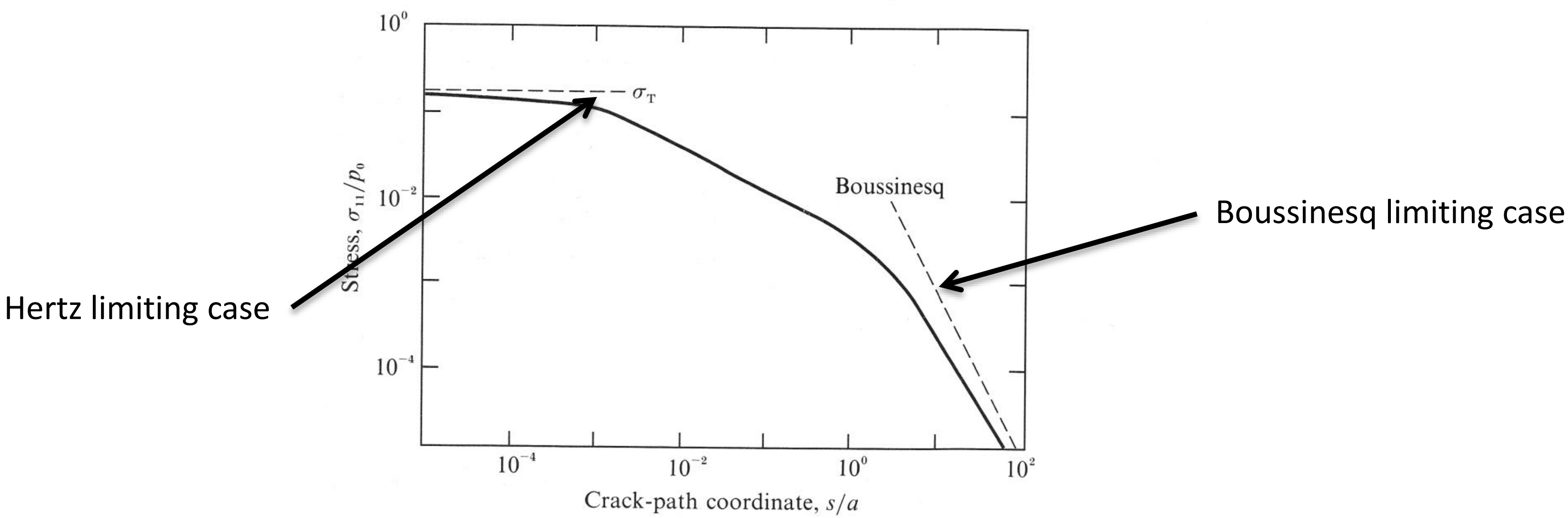
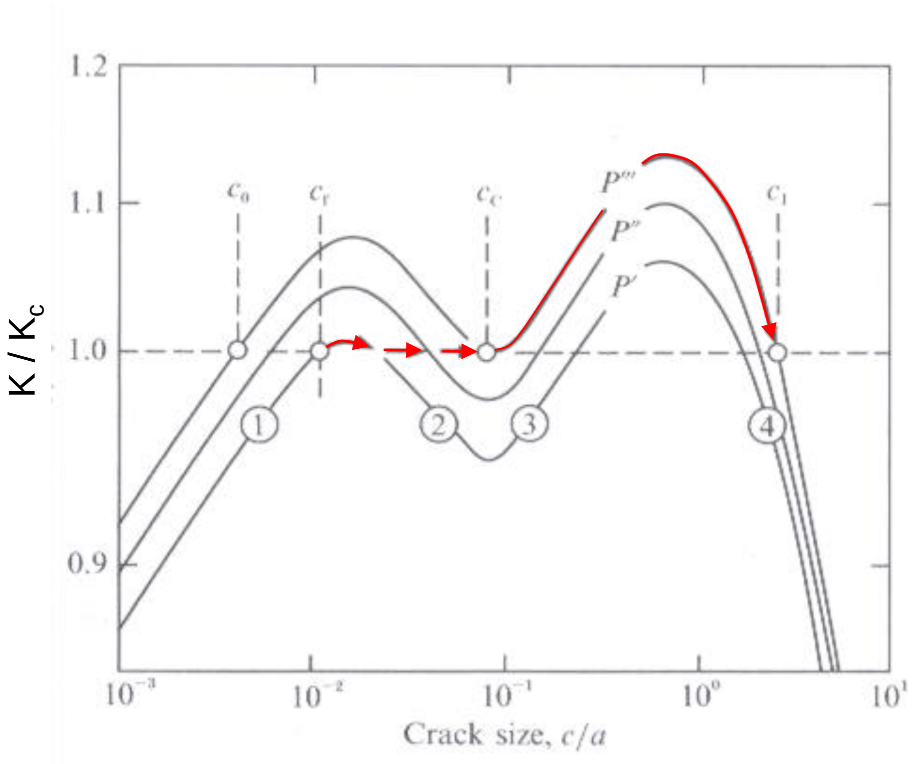


Fig. 8.30. Reduced plot of principal tensile stress σ_{11} vs distance downward along σ_{22}–σ_{33} stress trajectory surface in Hertzian field. Asymptotes (dashed lines) correspond to bounds of uniform field $\sigma_{11} = \sigma_T$ (Griffith flaw, $c \ll a$) and Boussinesq inverse-square field (Roesler cone, $c \gg a$). Note rapid stress falloff below surface, $\sigma_{11}/\sigma_T < 0.1$ at $s/a = 0.1$. Plots for $\beta = 1$, $\nu = \frac{1}{3}$. (After Frank, F. C. & Lawn, B. R. (1967) *Proc. Roy. Soc. Lond.* **A299** 291; Lawn, B. R. & Wilshaw, T. R. (1975) *J. Mater. Sci.* **10** 1049.)

Stability criterion for a Hertzian crack



Branchs(1) et (3) : instable propagation
Branchs (2) et (4) : stable propagation

Crack of length c_c :Hertzian cone formation

Critical load

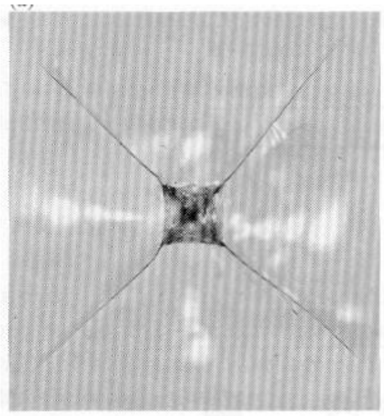
$$P_c \propto R\frac{K_c^2}{E^*}$$

Brittle materials : cone indentation

Vickers

• Plastic deformation

• Radial and median cracks



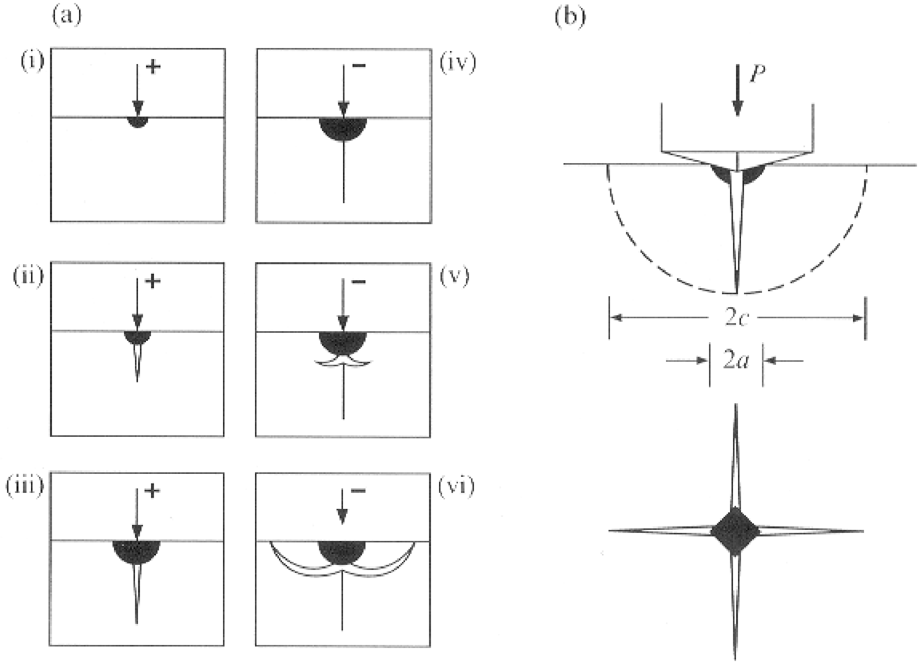


Fig. 8.7. Radial–median and lateral crack systems. (a) Evolution during complete loading (+) and unloading (−) cycle. Dark region denotes irreversible deformation zone. (b) Geometrical parameters of radial system.

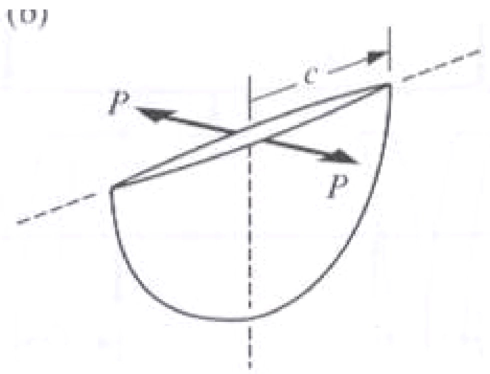
Residual stresses during unloading:

→Nucleation and propagation of median cracks

→ Surface propagation of radial cracks

Toughness measurements using indentation

$$\sigma_R Y\sqrt{a_c} = K_c$$



• Sphère / plan   c >> a P > P_c

$$\boxed{K_c = \chi \frac{P}{c^{3/2}}}$$

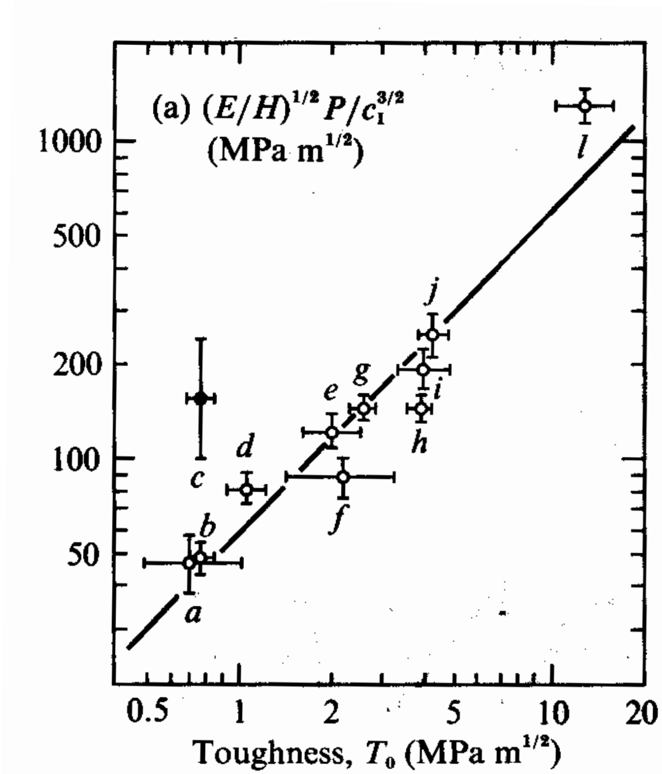
χ : non dimensional parameter dependent on Poisson's ratio

•Vickers

$$\boxed{K_c = \xi \left(\frac{E}{H}\right)^{1/2} \frac{P}{c^{3/2}}}$$

$$\xi = \xi_0 (cot\,\Phi)^{2/3}$$

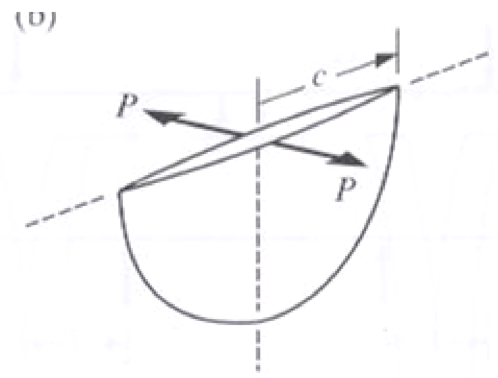
ξ_0 : non dimensional constant function of the strain distribution



Vickers indentation

Determination of fracture toughness from indentation testing

Half-penny crack $\sigma_R \sqrt{\pi a_c} = K_c$



• sphere on flat c >> a

$$K_c = \chi \frac{P}{c^{3/2}}$$

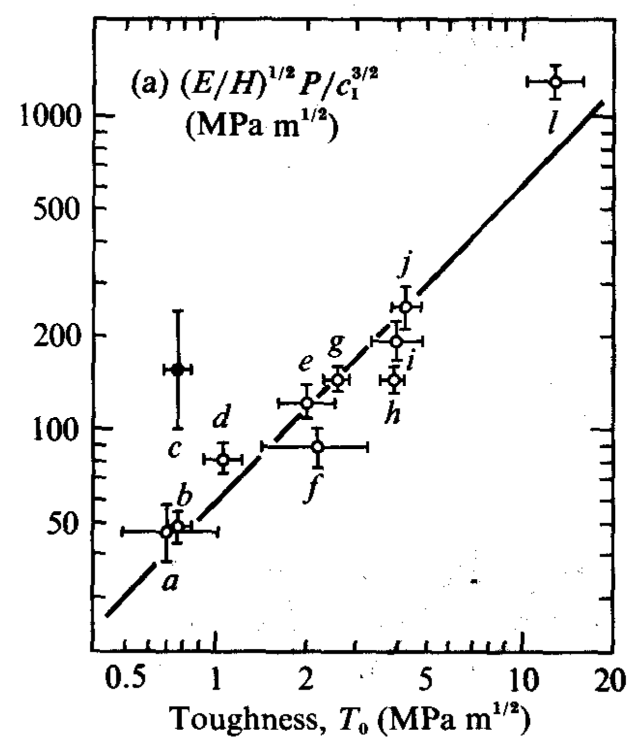
χ Non dimensional parameter depending on Poisson's ratio

• Vickers, cone…

$$K_c = \xi \left(\frac{E}{H}\right)^{1/2} \frac{P}{c^{3/2}}$$

$$\xi = \xi_0 (\cot \Phi)^{2/3}$$

ξ_0 Non dimensional parameter depending on indenter's geometry



Vickers indentation

Bibliography

• Fracture mechanics

B. Lawn
Fracture of brittle solids, Cambridge Solid State Science Series,Cambridge University Press, 1993

D. Maugis
Contact, adhesion and rupture of elastic solids, Springer, Solid State Sciences, 2000

• Fracture mechanics of polymer materials

J. G. Williams
Fracture mechanics of polymers, Halsted Press, New York, 1984