INTRODUCTION TO LINEAR ELASTIC FRACTURE MECHANICS
Theoretical strength of a perfectly ordered crystalline lattice

\[ \sigma = \sigma_{th} \sin \left( \frac{2\pi d}{\delta_0} \right) \]

\[ \sigma_{th} \approx \frac{E}{10} \]
Stress concentration at the vicinity of an elliptical hole

- Inglis calculation

Uniformly stressed plate

\[ \rho = \frac{b^2}{c} \]

\[ \frac{\sigma_c}{\sigma_a} \approx 2 \frac{c}{b} = 2 \sqrt{\frac{c}{\rho}} \]
**Crack in uniform tension**

Infinitely narrow elliptical cavity
Remote uniform tensile stress field $\sigma_A$

Unstable configuration

$$U (a) = -\pi a^2 \sigma_A^2 / E' + 4a\gamma$$
Cleavage by a wedge

- Obreimoff’s experiment (cleavage of mica)

\[ U(a) = \frac{Ed^3h^2}{8c^3} + 2a\gamma \]

Stable configuration
The Dupré work of adhesion:

\[ w = \gamma_1 + \gamma_2 - \gamma_{12} \]

Homogeneous solid \( w = 2\gamma \)

Equilibrium conditions!
No chemistry!
Condition for equilibrium fracture: the Griffith criterion

- Energy balance for crack extension

\[ dU = dU_M + dU_S = -GdA + wdA \]

where

\[ G = -\left( \frac{\partial U_E}{\partial A} + \frac{\partial U_P}{\partial A} \right) \]

\[ P = \left( \frac{\partial U_E}{\partial A} \right)_\delta \]

Strain energy release rate

\[ \frac{\partial U}{\partial A} = 0 \]

EQUILIBRIUM condition !!

\[ G = w \]

No dissipative processes !!

In most practical situations

\[ G_c \gg w \]

- Energy dissipation at the crack tip (plasticity, viscoelasticity) and/or in the bulk
- Dependence of actual fracture micro-mechanisms at the crack tip (process zone)

......
Irwin's approach: Linear elastic crack-tip fields

- The three modes of fracture
  
  I: opening mode  
  II: sliding mode (in plane shear)  
  III: tearing mode (out of plane shear)

- Irwin slit-crack tip in rectangular and polar coordinates
Stress-intensity factors $K$

\[
\sigma_{ij} = \frac{K}{\sqrt{2\pi r}} f_{ij}(\theta)
\]

\[
u_i = \frac{K}{2E} \sqrt{\frac{r}{2\pi}} f_i(\theta)
\]

\[K = \chi \sigma \sqrt{a}\]

$K = f \text{ (geometry, applied loading)}$

Mode I crack loading

\[
k = (3-v)/(1+v), \quad \nu^\prime = 0, \quad \nu^\prime = v. \quad \text{(plane stress)}
\]

\[
k = (3-4v), \quad \nu^\prime = v, \quad \nu^\prime = 0. \quad \text{(plane strain)}
\]

\[\begin{align*}
\frac{\sigma_{zz}}{(2\pi r)^{1/2}} &= \frac{K_1}{2} \left\{ \cos(\theta/2) \left[ 1 - \sin(\theta/2) \sin(3\theta/2) \right] \right\} \\
\sigma_{yy} &= \frac{K_1}{2} \left\{ \cos(\theta/2) \left[ 1 + \sin(\theta/2) \sin(3\theta/2) \right] \right\} \\
\sigma_{z\theta} &= \frac{K_1}{2} \left\{ \cos(\theta/2) \left[ \sin(\theta/2) \cos(\theta/2) \cos(3\theta/2) \right] \right\} \\
\sigma_{r\theta} &= \frac{K_1}{2} \left\{ \cos(\theta/2) \left[ 1 + \sin^2(\theta/2) \right] \right\} \\
\sigma_{rr} &= \frac{K_1}{2} \left\{ \cos^3(\theta/2) \right\} \\
\sigma_{\psi\psi} &= \frac{K_1}{2} \left\{ \sin(\theta/2) \cos(\theta/2) \cos(3\theta/2) \right\} \\
\sigma_{zz} &= \nu(\sigma_{zz} + \sigma_{yy}) - \nu^\prime (\sigma_{rr} + \sigma_{\psi\psi}) \\
\sigma_{yy} &= \sigma_{zz} - \sigma_{rr} = \sigma_{\psi\psi} = 0
\end{align*}\]
Equivalence of $G$ and $K$ parameters

- Extension and closure of the crack increment $CC'$

Fixed grip ($u=\text{const}$) calculation of the strain-energy release:

$$
\delta U_E = 2 \int_{a+\delta a}^{a} \frac{1}{2} (\sigma_{yy}u_y + \sigma_{xy}u_x + \sigma_{zy}u_z) \, dx
$$

$$
G = - \left( \frac{\partial U_E}{\partial a} \right)_{u} = \frac{K_{I}^2}{E'} + \frac{K_{II}^2}{E'} + \frac{K_{III}^2}{E'}(1 + \nu) / E
$$

plane strain: $E' = E/(1 - \nu^2)$; plane stress: $E' = E$
Cohesive zone models

Dugdale, Barenblatt

- Non linear cohesive stress $p(X,u)$ acting across the walls of the slip
- Superposition of the $K$ fields arising from the ‘external’ and ’internal contributions’
Physical origins of the cohesive zone

- Surface forces acting across the crack faces
- Plastic deformation at the crack tip (Dugdale model)
- Crack bridging in crazes (polymer fracture)
- ....
Barenblatt model: crack-opening profile

Hyp: \[ p(X, u)_{X \rightarrow c} \rightarrow 0 \]
\[ u(X)_{X \rightarrow c} \propto X^{3/2} \]
\[ \epsilon_{X \rightarrow c} = \frac{du}{dX} \rightarrow 0 \]
\[ u(X) \propto X^{1/2} \]
\[ \epsilon_{X \rightarrow c} \rightarrow \infty \]
J integral → holds for any reversible deformation response linear or non linear → path independent
Brittle materials: Hertzian fracture / I

**Sphere on flat contacts**

(ii) Crack initiation at the edge of the contact

(iii) Stable crack propagation

(iv) Instable crack propagation: formation of the Hertzian cone

(v) Stable propagation of the cone

(vi) Crack closing during unloading
Hertz contact: surface tractions

Maximum tensile stress:

\[
\sigma_R = \frac{1}{2} (1 - 2\nu) p_m \approx 0.1 p_m
\]

\( p_m = \text{mean contact pressure} \)
Point surface loading: sub-surface stress fields

Stress trajectories

Contours

Fig. 8.1. Boussinesq field, for principal normal stresses $\sigma_{11}$, $\sigma_{22}$ and $\sigma_{33}$.
(a) Stress trajectories, half-surface view (top) and side view (bottom).
(b) Contours, side view. Unit of contact stress is $p_0$, contact diameter $2a$ (arrow). Note sharp minimum in $\sigma_{11}(\phi)$ and zero in $\sigma_{22}(\phi)$, dashed lines. Plotted for $v = 0.25$. (See Johnson, K. L. (1985) Contact Mechanics. Cambridge University Press, Cambridge, Ch. 3.)
Critical load for the propagation of the Hertzian cone crack

- Stress-based approach:
  \[ \sigma_R = \frac{1}{2} (1 - 2\nu) p_m \quad \Rightarrow \quad P_c \propto R^2 \]

  Experimentally \[ P_c \propto R \]

- Linear elastic fracture mechanics analysis approach

  Non homogeneous stress field: the stress intensity factor is evolving as along the crack path

Fig. 8.30. Reduced plot of principal tensile stress \( \sigma_{11} \) vs distance downward along \( \sigma_{22} - \sigma_{33} \) stress trajectory surface in Hertzian field. Asymptotes (dashed lines) correspond to bounds of uniform field
\( \sigma_{11} = \sigma_T \) (Griffith flaw, \( c \ll a \)) and Boussinesq inverse-square field
(Roesler cone, \( c \gg a \)). Note rapid stress falloff below surface,
\( \sigma_{11}/\sigma_T < 0.1 \) at \( s/a = 0.1 \). Plots for \( \beta = 1 \), \( v = \frac{1}{3} \). (After Frank, F. C. &
**Stability criterion for a Hertzian crack**

Branches (1) et (3) : instable propagation

Branches (2) et (4) : stable propagation

Crack of length $c_c$ : Hertzian cone formation

Critical load

$$P_c \propto R \frac{K_c^2}{E^*}$$
Brittle materials: cone indentation

Vickers

- Plastic deformation

- Radial and median cracks

Residual stresses during unloading:

→ Nucleation and propagation of median cracks

→ Surface propagation of radial cracks

Fig. 8.7. Radial–median and lateral crack systems. (a) Evolution during complete loading (+) and unloading (−) cycle. Dark region denotes irreversible deformation zone. (b) Geometrical parameters of radial system.
Toughness measurements using indentation

\[ \sigma_R Y \sqrt{a_c} = K_c \]

- **Sphère / plan** \( c >> a \) \( P > P_c \)

\[ K_c = \chi \frac{P}{c^{3/2}} \]

\( \chi \): non dimensional parameter dependent on Poisson’s ratio

- **Vickers**

\[ K_c = \xi \left( \frac{E}{H} \right)^{1/2} \frac{P}{c^{3/2}} \]

\[ \xi = \xi_0 \left( \cot \Phi \right)^{2/3} \]

\( \xi_0 \): non dimensional constant function of the strain distribution

\( (E/H)^{1/2} P/c^{3/2} \) (MPa m\(^{1/2}\))

Toughness, \( T_0 \) (MPa m\(^{1/2}\))
Determination of fracture toughness from indentation testing

Half-penny crack \[ \sigma_R \sqrt{\pi a_c} = K_c \]

- sphere on flat \( c \gg a \)

\[ K_c = \chi \frac{P}{c^{3/2}} \]
\( \chi \): Non-dimensional parameter depending on Poisson’s ratio

- Vickers, cone...

\[ K_c = \xi \left( \frac{E}{H} \right)^{1/2} \frac{P}{c^{3/2}} \]
\( \xi = \xi_0 \left( \cot \Phi \right)^{2/3} \)
\( \xi_0 \): Non-dimensional parameter depending on indenter’s geometry

\[ (E/H)^{1/3} P/c^{3/3} \text{ (MPa m}^{1/3}) \]

Toughness, \( T_0 \) (MPa m^{1/3})
Bibliography

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